

Comment on "Characterization of an acoustic spherical cloak",
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The paper [1] considers a spherical cloak described by three radially varying acoustical quantities. For a given radial mass density in the cloak the question posed is whether the remaining two parameters, tangential density and compressibility, are uniquely determined. A method is proposed in [1] to solve this inverse question based upon the solution of a Riemann-Hilbert problem involving spectral properties of a one-dimensional inhomogeneous Schrödinger equation. However, no constructive examples of the solution procedure are given. This comment provides explicit solutions for any radial mass density that conforms with the requirements of transformation acoustics [2]. A valid form of the compatibility condition [1, eq. (22)] for $n = 0$ is derived.

1 Explicit solutions

The problem concerns a spherical shell V : $a < r < b$, the exterior infinite region V^e : $b < r$, and the interior V^i : $0 < r < a$. Both V^e and V^i are occupied by acoustic fluids of uniform mass density, compressibility and sound speed equal to $\{\rho_0, \gamma_0, c = (\rho_0\gamma_0)^{-1/2}\}$ and $\{\rho_1, \gamma_1, c_1 = (\rho_1\gamma_1)^{-1/2}\}$, respectively ([1] considers the case $\rho_1 = \rho_0, \gamma_1 = \gamma_0$). The cloaking domain V has radially varying parameters, radial density $\rho(r)$, tangential density $s(r)$ and compressibility $\gamma(r)$. The objective is to ensure zero scattering for plane wave incidence. Time harmonic dependence $e^{-i\omega t}$ is assumed. Let u denote the acoustic pressure field, then in V^e $u = u^e(\mathbf{r}) \equiv e^{i\mathbf{k}\cdot\mathbf{r}}$ where $\mathbf{k} = k\hat{\mathbf{k}}$ is the wavenumber vector of magnitude $k = \omega/c$.

1.1 $s(r)$ and $\gamma(r)$ in terms of $\rho(r)$

Consider eq. (19) of [1], replacing f_n^m by f_n since the solutions are independent of m ,

$$\frac{d}{dr} \left(\frac{r^2}{\rho(r)} \frac{df_n(r)}{dr} \right) + \left(r^2 \gamma(r) \omega^2 - \frac{n(n+1)}{s(r)} \right) f_n(r) = 0, \quad a < r < b. \quad (1)$$

For a given positive radial density $\rho(r)$, define

$$s(r) = \frac{\rho_0^2 r^2}{\rho(r) R^2(r)}, \quad \gamma(r) = \frac{\rho_0^3 \gamma_0}{\rho(r) s^2(r)} \quad (2)$$

where

$$R(r) = \left(\frac{1}{b} + \int_r^b \frac{\rho(t)}{\rho_0 t^2} dt \right)^{-1}. \quad (3)$$

The dependence $r \rightarrow R(r)$ is one-to-one since $\rho(r)$ is positive and hence R can be considered a mapping or transformation. Equation (1) becomes

$$\frac{d}{dR} \left(R^2 \frac{dF_n(R)}{dR} \right) + (k^2 R^2 - n(n+1)) F_n(R) = 0, \quad R(a) < R < b \quad (4)$$

where

$$F_n(R) = f_n(r). \quad (5)$$

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The unique solution of (1) satisfying the boundary conditions eqs. (20) and (21) of [1]

$$f_n(b) = j_n(kb), \quad \frac{1}{\rho(b)} f'_n(b) = \frac{k}{\rho_0} j'_n(kb) \quad (6)$$

is therefore

$$f_n(r) = j_n(kR(r)). \quad (7)$$

Example 1. The radial density of [3] is

$$\rho(r) = \frac{b-a}{b} \left(\frac{r}{r-a} \right)^2 \rho_0, \quad (8)$$

for which eqs. (2) and (3) give

$$R(r) = \frac{r-a}{b-a} b, \quad s(r) = \frac{b-a}{b} \rho_0, \quad \gamma(r) = \left(\frac{b}{b-a} \right)^3 \left(\frac{r-a}{r} \right)^2 \gamma_0. \quad (9)$$

Note that $\gamma(r)$ follows from (2)₂, i.e. the identity $\rho(r)s^2(r)\gamma(r) = \rho_0^3\gamma_0$. In further examples $\gamma(r)$ is not given explicitly.

Example 2. The linear near-cloak of [4] corresponds to

$$\rho(r) = \frac{b-a}{b-\delta} \left(\frac{r}{r - \left(\frac{a-\delta}{b-\delta} \right) b} \right)^2 \rho_0, \quad (10)$$

for which eqs. (2) and (3) yield

$$R(r) = \frac{r-a}{b-a} b + \frac{b-r}{b-a} \delta, \quad s(r) = \frac{b-a}{b-\delta} \rho_0. \quad (11)$$

This example reduces to Example 1 if $\delta = 0$.

Example 3. Constant radial density $\rho(r) = \rho_c$ yields

$$R(r) = \left(\frac{1}{b} + \frac{\rho_c}{\rho_0} \left(\frac{1}{r} - \frac{1}{b} \right) \right)^{-1}, \quad s(r) = \frac{r^2 \rho_0^2}{R^2(r) \rho_c}. \quad (12)$$

Example 4. In general, for a given function $R(r)$ [2, eq. (2.16)],

$$\rho(r) = \frac{\rho_0 r^2}{R^2(r)} R'(r), \quad s(r) = \frac{\rho_0}{R'(r)}, \quad \gamma(r) = \frac{R^2(r)}{r^2} R'(r) \gamma_0. \quad (13)$$

The parameters $\{\rho(r), s(r), \gamma(r)\}$ correspond respectively to $\{1/K_r(r), 1/K_\perp(r), \rho(r)\}$ of [5] where other examples are considered.

2 Compatibility condition

The solution $f_n(r)$, $n \geq 0$, must satisfy a compatibility condition at $r = a$ if the interior region is to be cloaked. We revisit this because of the more general problem considered here of an interior region with different properties than the exterior, and also because the compatibility condition [1, eq. (22)] is not valid for $n = 0$. A corrected compatibility condition for $n = 0$ will be derived.

The pressure in V^i is $u^i(\mathbf{r})$ which satisfies the Helmholtz equation [1, eq. (2)]

$$\Delta u^i(\mathbf{r}) + k_1^2 u^i(\mathbf{r}) = 0, \quad \mathbf{r} \in V^i \quad (14)$$

where $k_1 = \omega/c_1$. The particle velocity follows from [1, eq. (7)] as

$$\mathbf{v} = (i\omega\rho_1)^{-1} \nabla u^i, \quad \mathbf{r} \in V^i. \quad (15)$$

The solutions in the core and the shell are [1, eqs. (17), (18)]

$$\left. \begin{matrix} u^i(\mathbf{r}) \\ u(\mathbf{r}) \end{matrix} \right\} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{m=n} i^n Y_n^m(\hat{\mathbf{r}}) Y_n^m(\hat{\mathbf{k}})^* \times \begin{cases} A_n^m j_n(k_1 r), & \mathbf{r} \in V^i, \\ f_n^m(r), & \mathbf{r} \in V. \end{cases} \quad (16)$$

The continuity conditions for pressure and radial velocity at the interface $r = a$ [1, eqs. (8), (9)] therefore become

$$\begin{aligned} f_n^m(a) &= A_n^m j_n(k_1 a), \\ \frac{1}{\rho(a)} f_n^{m'}(a) &= \frac{k_1}{\rho_1} A_n^m j_n'(k_1 a). \end{aligned} \quad (17)$$

Eliminating A_n^m and dropping the redundant superscript m yields the compatibility condition

$$k_1 \rho(a) j_n'(k_1 a) f_n(a) = \rho_1 j_n(k_1 a) f_n'(a) \quad (18)$$

which is the generalized form of [1, eq. (22)].

The compatibility condition is satisfied by the solution (7) iff

$$j_n'(k_1 a) j_n(kR(a)) - \frac{R^2(a)}{a^2} \frac{\rho_1 c_1}{\rho_0 c} j_n(k_1 a) j_n'(kR(a)) = 0. \quad (19)$$

This holds $\forall n \geq 1$ if

$$R(a) = 0 \quad (20)$$

which is the condition expected from transformation acoustics [2]. Using the identity (20) the $n = 0$ compatibility condition (19) reduces to

$$-j_1(k_1 a) = 0, \quad (21)$$

which only holds if $k_1 a$ is a zero of j_1 . Hence, the $n = 0$ compatibility condition does not appear to be correct since it is not satisfied even for vanishing $R(a)$, which is sufficient according to transformation acoustics. The same inconsistency applies to the compatibility condition [1, eq. (22)] for the radial density considered in Example 1 of [1], i.e. $\rho(r)$ of [1, eq. (27)] which is the same as $\rho(r)$ of eq. (8) above. Thus, using this $\rho(r)$, along with $f_n(r) = j_n(kb(r-a)/(b-a))$ [1, eq. (32)] in eq. (18) (or equivalently [1, eq. (22)]) for $n = 0$ yields zero on the right hand side and infinity on the left. Furthermore, this indicates that the value of A_0 in [1, eq. (34)] is suspect. The proper form of the $n = 0$ compatibility condition and the correct value of A_0 are examined next.

2.1 A compatibility condition for all $n \geq 0$

In order to derive a consistent compatibility condition that holds for all values of $n \geq 0$ we introduce waves going both ways in V and V^e . The solution in V^i is still given by (16), while the solutions in the cloak and exterior domains become

$$\left. \begin{matrix} u^e(\mathbf{r}) \\ u(\mathbf{r}) \end{matrix} \right\} = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{m=n} i^n Y_n^m(\hat{\mathbf{r}}) Y_n^m(\hat{\mathbf{k}})^* \times \begin{cases} (j_n(kr) + B_n^m h_n(kr)), & \mathbf{r} \in V^e, \\ (j_n(kR(r)) + B_n^m h_n(kR(r))), & \mathbf{r} \in V. \end{cases} \quad (22)$$

where $h_n = h_n^{(1)}$ is the spherical Hankel function of the first kind. Continuity of pressure and radial velocity at $r = b$ is automatically satisfied. The interface conditions at $r = a$ become (dropping the superscript m)

$$\begin{aligned} j_n(kR(a)) + B_n h_n(kR(a)) &= A_n j_n(k_1 a), \\ \frac{\rho_1 c_1}{\rho_0 c} \epsilon^2 (j'_n(kR(a)) + B_n h'_n(kR(a))) &= A_n j'_n(k_1 a), \end{aligned} \quad (23)$$

with solutions,

$$A_n = \frac{-i \frac{\rho_1 c_1}{\rho_0 c} (ka)^{-2}}{h_n(kR(a)) j'_n(k_1 a) - \frac{\rho_1 c_1}{\rho_0 c} \frac{R^2(a)}{a^2} h'_n(kR(a)) j_n(k_1 a)}, \quad (24)$$

$$B_n = - \left(\frac{j_n(kR(a)) j'_n(k_1 a) - \frac{\rho_1 c_1}{\rho_0 c} \frac{R^2(a)}{a^2} j'_n(kR(a)) j_n(k_1 a)}{h_n(kR(a)) j'_n(k_1 a) - \frac{\rho_1 c_1}{\rho_0 c} \frac{R^2(a)}{a^2} h'_n(kR(a)) j_n(k_1 a)} \right) \quad (25)$$

where $j_n(x)h'_n(x) - j'_n(x)h_n(x) = ix^{-2}$ has been used. As a compatibility condition we require

$$B_n = 0 \quad (26)$$

which is satisfied by $R(a) = 0$, i.e. eq. (20), for all $n \geq 0$. Furthermore, taking the limit of (24) as $R(a) \rightarrow 0$ yields

$$A_n = 0, \quad n \geq 0. \quad (27)$$

In summary, the compatibility condition (26), which is valid for all $n \geq 0$, replaces the condition (19) (and [1, eq. (22)]), and eq. (27) corrects eq. (34) of [1].

3 Conclusion

There is an infinite set of radial density functions $\rho(r)$ for which the tangential density $s(r)$ and compressibility $\gamma(r)$ are given by eqs. (2) and (3). These solutions require that $R(a) = 0$ which in turn means that $\rho(r)$ must be singular as $r \downarrow a$. Define the radial "mass" of the cloak enclosed between r and b ,

$$m(r) = 4\pi \int_r^b \rho(r) r^2 dr, \quad (28)$$

which can be expressed as

$$m(r) = 4\pi \rho_0 a^4 \left(\frac{1}{R(r)} - \frac{1}{b} \right) + 4\pi \int_r^b \left(\frac{r^4 - a^4}{r^2} \right) \rho(r) dr. \quad (29)$$

The latter integral is positive while the term $1/R(r) \rightarrow \infty$ as $r \rightarrow a$. Hence, the total radial mass of the cloak must be infinite. This conundrum was noted in [2] where an alternative solution involving anisotropic stiffness and isotropic density was proposed, and was shown to have finite mass equal to the mass of fluid inside a sphere of radius b .

The question regarding [1] is whether other solutions exist for the positive triplet $\rho(r)$, $s(r)$, $\gamma(r)$, specifically solutions for which $\rho(a)$ is finite. Intuition suggests that the answer is no, that the only solutions correspond to the transformation function $R(r)$ mapping the inner boundary $r = a$ to the origin. A counterexample would not only be very interesting, but would provide new directions for research: one can only hope that the algorithm developed in [1] can provide some insight.

References

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